

ON FINITE TYPE FLAT MODULES

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ABSTRACT. In this article, the finitely generated flat modules of a commutative ring are studied both from an algebraic point of view and from a topological point of view. Then, as an application, some major results in the literature on the projectivity of f.g. flat modules are vastly generalized.

1. INTRODUCTION

There are numerous problems in arithmetic and algebraic geometry which are related somehow to the projective modules. Hence studying the projectivity of modules has found a valuable position in mathematics specially in both commutative and non-commutative algebra. Amongst them, studying the projectivity of f.g. flat modules has a special importance since such modules are locally free, see Corollary 3.5. Note that in general there are f.g. flat modules which are not necessarily projective, see Example 5.6 also see [3, Tag 00NY]. Therefore, following [8], it makes sense to define a ring R as an S-ring (“S” refers to Sakhajev) if every f.g. flat R -module is R -projective. Note that the notion of “S-ring” is the same as “A(0) ring” in [1]. Studying the projectivity of f.g. flat modules, in particular exploring S-rings, has been the mainstream of researches in the scientific literature over the years and continuously mathematicians show increasing interest to this topic, see e.g. [1], [2], [4], [5], [6, §4E], [8], [13], [14].

The main purpose of the present article is to investigate the projectivity of f.g. flat modules by applying both an algebraic method (mostly based on the exterior powers of a module) and a topological method (using the spectral topologies of the prime spectrum). The important outcome of this study is that some major results in the literature on the projectivity of f.g. flat modules are re-proved directly (e.g. without using the homological methods) and at the same time

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most of them are vastly generalized. Because of the plenty of the generalized results we omitted to mention them at here and we refer to each of them at its appropriate place along the article. This study also reveals that the class of S-rings is considerably broad.

In order to clarify the efficiency of the topological methods in this study, for instance we point out that Vasconcelos in [13, Corollary 1.7], shows that a ring R is an S-ring if and only if the cyclic flat R -modules are projective. In partial of the present article, this result not only is re-proved but also, by using the topological methods, its condition is significantly weakened. In fact, in Theorem 4.3, it is shown that a ring R is an S-ring if and only if every patch closed subset of $\text{Spec}(R)$ which is stable under the generalization and specialization is patch open. Then, under this weakened condition, Theorem 5.1 is obtained which significantly generalizes some previous results in the literature on the projectivity of f.g. flat modules.

For reading the present article having a reasonable knowledge from the exterior powers of a module is necessary. We refer the reader to [12, §6] for studying of the topic up to level which is needed in this article. Familiarity with some topological notions are also needed which we briefly recall them at here. A subset of $\text{Spec}(R)$ is said to be a double-closed if it is closed w.r.t. the both flat and Zariski topologies. By [10, Theorem 3.11], the double-closed subsets of $\text{Spec}(R)$ are precisely the patch closed subsets which are stable under the generalization and specialization. We use f.g. in place of finitely generated. All of the rings which are discussed in this article are commutative. The titles of the sections should be sufficiently explanatory.

2. PRELIMINARIES

Lemma 2.1. *Let M be a f.g. R -module, let $I = \text{Ann}_R(M)$ and let S be a multiplicative subset of R . Then $S^{-1}I = \text{Ann}_{S^{-1}R}(S^{-1}M)$.*

Proof. Easy. \square

Lemma 2.2. *Let M be a f.g. R -module and let I be an ideal of R such that $IM = M$. Then there is an element $a \in I$ such that $(1+a)M = 0$.*

Proof. It is well-known, see [3, Tag 00DW]. \square

3. A REMARK ON FLAT MODULES AND THE INVARIANT FACTORS

Lemma 3.1. *The annihilator of a f.g. projective module is generated by an idempotent element.*

Proof. Let M be a R -module, let $I = \text{Ann}_R(M)$ and let J be the ideal of R which is generated by the elements $f(m)$ where $f : M \rightarrow R$ is a R -linear map and $m \in M$. Clearly $IJ = 0$. Consider a free R -module F with basis $\{e_i\}$ and an onto R -linear map $\psi : F \rightarrow M$. For each i there is a R -linear map $h_i : F \rightarrow R$ such that $h_i(e_j) = \delta_{i,j}$. If M is R -projective then there exists a R -linear map $\varphi : M \rightarrow F$ such that $\psi \circ \varphi$ is the identity map. Put $f_i = h_i \circ \varphi$ for all i . Then for each $m \in M$ we may write $m = \sum_i f_i(m)\psi(e_i)$ where $f_i(m) = 0$ for all but a finite number of indices i . This implies that $JM = M$. If moreover M is a f.g. R -module then, by Lemma 2.2, we may find an element $b \in J$ such that $1 + b \in I$. Let $a = 1 + b$. Then clearly $a = a^2$ and $I = Ra$. \square

Remark 3.2. Here we show that in a flat module both of the scalars and vectors involved in a linear relation have very peculiar properties. Let M be a R -module and consider a linear relation $\sum_{i=1}^n a_i x_i = 0$ in M where $a_i \in R$ and $x_i \in M$ for all i . Let $I = \langle a_1, \dots, a_n \rangle$ and consider the map $\psi : R^n \rightarrow I$ which maps each n -tuple (r_1, \dots, r_n) into $\sum_{i=1}^n r_i a_i$. Then we get the following exact sequence of R -modules

$$K \otimes_R M \xrightarrow{i \otimes 1} R^n \otimes_R M \xrightarrow{\psi \otimes 1} I \otimes_R M \quad \text{where } K = \text{Ker } \psi.$$

If moreover M is R -flat then $\sum_{i=1}^n \epsilon_i \otimes x_i \in \text{Ker}(\psi \otimes 1)$ where $\epsilon_i = (\delta_{i,k})_{1 \leq k \leq n}$ for all i . Because by the flatness of M , $I \otimes_R M$ is canonically isomorphic to IM . Therefore there exist a natural number $m \geq 1$ and also elements $s_j = (r_{1,j}, \dots, r_{n,j}) \in K$ and $y_j \in M$ with $1 \leq j \leq m$ such that $(i \otimes 1)(\sum_{j=1}^m s_j \otimes y_j) = \sum_{i=1}^n \epsilon_i \otimes x_i$. Now by applying the canonical isomorphism $R^n \otimes_R M \rightarrow M^n$ which maps each pure tensor $(r_1, \dots, r_n) \otimes x$ into $(r_1 x, \dots, r_n x)$ we obtain that $x_i = \sum_{j=1}^m r_{i,j} y_j$ for all i . Moreover for

each j , $\sum_{i=1}^n r_{i,j}a_i = 0$ since $\psi(s_j) = 0$.

Under the light of Remark 3.2, the following result is obtained.

Theorem 3.3. *Let (R, \mathfrak{m}) be a local ring and let M be a flat R -module. Let S be a subset of M such that its image under the canonical map $M \rightarrow M/\mathfrak{m}M$ is linearly independent over $k = R/\mathfrak{m}$. Then S is linearly independent over R .*

Proof. Suppose $\sum_{i=1}^n a_i x_i = 0$ where $a_i \in R$ and $\{x_1, \dots, x_n\} \subseteq S$. To prove the assertion we shall use an induction argument on n . If $n = 1$ then by Remark 3.2, there are elements $z_1, \dots, z_d \in M$ and $r_1, \dots, r_d \in R$ such that $x_1 = \sum_{j=1}^d r_j z_j$ and $r_j a_1 = 0$ for all j . By the hypotheses, $x_1 \notin \mathfrak{m}M$. Therefore $r_j \notin \mathfrak{m}$ for some j . It follows that $a_1 = 0$. Now let $n > 1$. Again by Remark 3.2, there are elements $y_1, \dots, y_m \in M$ and $r_{i,j} \in R$ such that $x_i = \sum_{j=1}^m r_{i,j} y_j$ and $\sum_{i=1}^n r_{i,j} a_i = 0$ for all j . There is some j such that $r_{n,j} \notin \mathfrak{m}$ since $x_n \notin \mathfrak{m}M$. It follows that $a_n = \sum_{i=1}^{n-1} r_{n,j}^{-1} r_{i,j} a_i$. Then we get $\sum_{i=1}^{n-1} a_i (x_i + r_{n,j}^{-1} r_{i,j} x_n) = 0$. Let $c_i = r_{n,j}^{-1} r_{i,j}$. Note that the image of $\{x_i + c_i x_n : 1 \leq i \leq n-1\}$ under the canonical map $M \rightarrow M/\mathfrak{m}M$ is linearly independent (because if $\{x_1, \dots, x_n\}$ is a linearly independent subset of a module then $\{x_i + r_i x_n : 1 \leq i \leq n-1\}$ is also a linearly independent subset where the r_i are arbitrary scalars). Therefore, by the induction hypothesis, $a_i = 0$ for all $1 \leq i \leq n-1$. This also implies that $a_n = 0$. \square

Corollary 3.4. *Let (R, \mathfrak{m}) be a local ring and let M be a flat R -module. Then there is a free R -submodule F of M such that $M = F + \mathfrak{m}M$. In particular, if either M/F is finitely generated or the maximal ideal is nilpotent then M is a free R -module.*

Proof. Every vector space has a basis. So let $\{x_i + \mathfrak{m}M : i \in I\}$ be a k -basis of $M/\mathfrak{m}M$ where $k = R/\mathfrak{m}$. By Theorem 3.3, $F = \sum_{i \in I} R x_i$ is a free R -module. Clearly $M = F + \mathfrak{m}M$. If M/F is finitely generated then by the Nakayama lemma (Lemma 2.2), $M = F$. If \mathfrak{m} is nilpotent

then there is a natural number $n \geq 1$ such that $\mathfrak{m}^n = 0$. It follows that $M/F = \mathfrak{m}^n(M/F) = 0$. \square

As an immediate consequence of the above corollary we obtain the following result which plays a major role in this article:

Corollary 3.5. *Every f.g. flat module over a local ring is free.* \square

As the first application of the above result we get:

Lemma 3.6. *The annihilator of a f.g. flat module is an idempotent ideal.*

Proof. Let M be a f.g. flat module over a ring R . Let $I = \text{Ann}_R(M)$. Let \mathfrak{p} be a prime ideal of R . By Lemma 2.1, $I_{\mathfrak{p}} = \text{Ann}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$. By Corollary 3.5, $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$ -module. Therefore $I_{\mathfrak{p}}$ is either the whole localization or the zero ideal. If $I_{\mathfrak{p}} = 0$ then $(I^2)_{\mathfrak{p}} = 0$ since $I^2 \subseteq I$. But if $I_{\mathfrak{p}} = R_{\mathfrak{p}}$ then there is an element $a \in I \setminus \mathfrak{p}$. Clearly $a^2 \in I^2 \setminus \mathfrak{p}$ and so $(I^2)_{\mathfrak{p}} = R_{\mathfrak{p}}$. Therefore $I = I^2$. \square

If M is a R -module then the n -th invariant factor of M , $I_n(M)$, is defined as the annihilator of the n -th exterior power of M , i.e., $I_n(M) = \text{Ann}_R(\Lambda^n(M))$. We have then:

Lemma 3.7. *The invariant factors of a f.g. flat module are idempotent ideals.*

Proof. Let M be a f.g. flat R -module. By [12, Lemmata 6.3 and 6.6], $\Lambda^n(M)$ is a f.g. flat R -module. Thus, by Lemma 3.6, $I_n(M)$ is an idempotent ideal. \square

Theorem 3.8. *Let M be a f.g. flat R -module. Then the following conditions are equivalent.*

- (i) *M is R -projective.*
- (ii) *The invariant factors of M are f.g. ideals.*
- (iii) *The rank map of M is Zariski continuous.*

Proof. (i) \Rightarrow (ii) : By [12, Lemmata 6.3 and 6.6], $\Lambda^n(M)$ is a f.g. projective R -module and so by Lemma 3.1, $I_n(M)$ is a principal ideal.

(ii) \Rightarrow (iii) : By Lemmata 2.2 and 3.7, there exists some $a \in I_n(M)$ such that $(1 - a)I_n(M) = 0$. Then clearly $a = a^2$ and $I_n(M) = Ra$. This, in particular, implies that the rank map of M , see [12, Remark 2.3], is Zariski continuous. Because, by [12, Lemma 4.1], $\psi^{-1}(\{n\}) = \text{Supp } N \cap (\text{Spec}(R) \setminus \text{Supp } N')$ where $N = \Lambda^n(M)$ and $N' = \Lambda^{n+1}(M)$. But $\text{Supp } N = D(1 - a)$. Moreover, $\text{Supp } N' = V(I_{n+1}(M))$ since, by [12, Lemma 6.3], N' is a f.g. R -module.
 (iii) \Rightarrow (i) : Apply Corollary 3.5 and [3, Tag 00NX]. \square

Now, as the first main result, we obtain the following result which vastly generalizes [2, Theorem 1].

Theorem 3.9. *Let $A \subseteq B$ be an extension of rings and let M be a f.g. flat A -module. If $M \otimes_A B$ is B -projective then M is A -projective.*

Proof. First we shall prove that $I = \text{Ann}_A(M)$ is a principal ideal. Let $L = \text{Ann}_B(N)$ where $N = M \otimes_A B$. We claim that $IB = L$. Let \mathfrak{q} be a prime ideal of B . Clearly N is a f.g. B -module. Thus, by Corollary 3.5, $L_{\mathfrak{q}}$ is either the whole localization or the zero ideal. If $L_{\mathfrak{q}} = 0$ then $(IB)_{\mathfrak{q}} = 0$ since $IB \subseteq L$. But if $L_{\mathfrak{q}} = B_{\mathfrak{q}}$ then L is not contained in \mathfrak{q} and so $N_{\mathfrak{q}} = 0$. Again by Corollary 3.5, $I_{\mathfrak{p}}$ is either the whole localization or the zero ideal where $\mathfrak{p} = A \cap \mathfrak{q}$. If $I_{\mathfrak{p}} = 0$ then $M_{\mathfrak{p}} \neq 0$ and so, by Corollary 3.5, $M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{q}} \neq 0$. But $M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{q}}$ is isomorphic to $N_{\mathfrak{q}}$. This is a contradiction. Therefore $I_{\mathfrak{p}} = A_{\mathfrak{p}}$. It follows that $(IB)_{\mathfrak{q}} = B_{\mathfrak{q}}$. This establishes the claim. By Lemma 3.1, there is an idempotent $e \in B$ such that $IB = Be$. Let $J = B(1 - e) \cap A$. Clearly $IJ = 0$. We have $I + J = A$. If not, then there exists a prime ideal \mathfrak{p} of A such that $I + J \subseteq \mathfrak{p}$. Thus, by Corollary 3.5, $I_{\mathfrak{p}} = 0$. Therefore the extension of IB under the canonical map $B \rightarrow B \otimes_A A_{\mathfrak{p}}$ is zero. Thus there exists an element $s \in A \setminus \mathfrak{p}$ such that $se = 0$ and so $s = s(1 - e)$. Hence $s \in J$. But this is a contradiction. Therefore $I + J = A$. It follows that there is an element $c \in I$ such that $c = c^2$ and $I = Ac$. Now, let $n \geq 1$. By [12, Lemmata 6.3 and 6.6], $\Lambda^n(M)$ is a f.g. flat A -module. Moreover, $\Lambda^n(M) \otimes_A B$ is B -projective since, by [12, Theorem 6.7], it is isomorphic to $\Lambda_B^n(M \otimes_A B)$ and by [12, Lemma 6.6], $\Lambda_B^n(M \otimes_A B)$ is B -projective. Thus, by what we have just proved, $I_n(M)$ is a principal ideal. Hence, by Theorem 3.8, M is A -projective. \square

Lemma 3.10. *Let M be a f.g. flat R -module and let J be an ideal of R . Let $I = \text{Ann}_R(M)$ and $L = \text{Ann}_R(M/JM)$. Then $L = I + J$.*

Proof. Clearly $I + J \subseteq L$. Let \mathfrak{p} be a prime ideal of R . By Lemma 2.1, $I_{\mathfrak{p}} = \text{Ann}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$. Thus, by Corollary 3.5, $I_{\mathfrak{p}}$ is either the whole localization or the null ideal for all primes \mathfrak{p} . If $I_{\mathfrak{p}} = R_{\mathfrak{p}}$ then $(I + J)_{\mathfrak{p}} = L_{\mathfrak{p}} = R_{\mathfrak{p}}$ since $I \subseteq I + J \subseteq L$. But if $I_{\mathfrak{p}} = 0$ then $M_{\mathfrak{p}} \neq 0$ and so $\text{Ann}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}/J_{\mathfrak{p}}M_{\mathfrak{p}}) = J_{\mathfrak{p}}$ (recall that if F is a non-zero free R -module then $\text{Ann}_R(F/JF) = J$). On the other hand, by Lemma 2.1, $L_{\mathfrak{p}} = \text{Ann}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}/J_{\mathfrak{p}}M_{\mathfrak{p}})$. Thus $(I + J)_{\mathfrak{p}} = J_{\mathfrak{p}} = L_{\mathfrak{p}}$. Hence $L = I + J$. \square

The following result generalizes [13, Theorem 2.1].

Theorem 3.11. *Let M be a f.g. flat R -module and let J be an ideal of R which is contained in the radical Jacobson of R . If M/JM is R/J -projective then M is R -projective.*

Proof. First we shall prove that $I = \text{Ann}_R(M)$ is a principal ideal. By Lemma 3.10, $L = I + J$. Also, by Lemma 3.1, $\text{Ann}_{R/J}(M/JM) = L/J$ is a principal ideal. This implies that $I = Rx + I \cap J$ for some $x \in R$ since $L/J = I + J/J$ is canonically isomorphic to $I/(I \cap J)$. But $I = Rx$. Because let \mathfrak{m} be a maximal ideal of R . By Corollary 3.5, $I_{\mathfrak{m}}$ is either the whole localization or the zero ideal. If $I_{\mathfrak{m}} = 0$ then $(Rx)_{\mathfrak{m}} = 0$ since $Rx \subseteq I$. But if $I_{\mathfrak{m}} = R_{\mathfrak{m}}$ then I is not contained in \mathfrak{m} . Thus Rx is also not contained in \mathfrak{m} since $I \cap J \subseteq J \subseteq \mathfrak{m}$. Hence $(Rx)_{\mathfrak{m}} = R_{\mathfrak{m}}$. Therefore $I = Rx$. Now let $n \geq 1$ and let $N = \Lambda^n(M)$. Then N/JN is R/J -projective. Because, by [12, Theorem 6.7], N/JN as R/J -module is isomorphic to $\Lambda_{R/J}^n(M/JM)$ and by [12, Lemma 6.6], $\Lambda_{R/J}^n(M/JM)$ is R/J -projective. By [12, Lemmata 6.3 and 6.6], N is a f.g. flat R -module. Therefore, by what we have just proved, we conclude that $I_n(M) = \text{Ann}_R(N)$ is a principal ideal. Thus the invariant factors of M are f.g. ideals and so by Theorem 3.8, M is R -projective. \square

4. TOPOLOGICAL ASPECTS OF F.G. FLAT MODULES

Lemma 4.1. *Let I be an ideal of a ring R . Then R/I is R -flat if and only if $\text{Ann}_R(a) + I = R$ for all $a \in I$. Moreover, in this case, for each finite subset $\{a_1, \dots, a_n\}$ of elements of I then there is some $c \in I$ such*

that $a_i = a_i c$ for all i .

Proof. First assume that R/I is R -flat. Suppose there is some $a \in I$ such that $\text{Ann}_R(a) + I \neq R$. Thus there exists a prime \mathfrak{p} of R such that $\text{Ann}_R(a) + I \subseteq \mathfrak{p}$. Therefore, by Corollary 3.5, $I_{\mathfrak{p}} = 0$ and so there is an element $s \in R \setminus \mathfrak{p}$ such that $sa = 0$. But this is a contradiction and we win. The converse is relatively easy, see [11, Corollary 3.4]. To see the last part of the assertion, note that for each pair (a, a') of elements of I then there are $b, b' \in I$ such that $a = ab$ and $a' = a'b'$. Clearly $c = b + b' - bb' \in I$, $a = ac$ and $a' = a'c$. \square

Fix a ring R and let E be a subset of $\text{Spec}(R)$. Consider $\mathcal{F}(E) = \bigcup_{\mathfrak{p} \in E} \Lambda(\mathfrak{p})$ where $\Lambda(\mathfrak{p}) = \{\mathfrak{q} \in \text{Spec}(R) : \mathfrak{q} \subseteq \mathfrak{p}\}$. We call $\mathcal{F}(E)$ the *down cones* of E . Similarly, consider $\mathcal{Z}(E) = \bigcup_{\mathfrak{p} \in E} V(\mathfrak{p})$. It is called the *up cones* of E . Then we have:

Theorem 4.2. (i) If E is Zariski closed then $\mathcal{F}(E)$ is flat closed.
(ii) If E is patch closed then $\mathcal{Z}(E)$ is Zariski closed.
(iii) If E is stable under the generalization and $E = V(I)$ for some ideal I of R then R/J is R -flat where J is the kernel of the canonical map $R \rightarrow S^{-1}R$ with $S = 1 + I$. Moreover, $E = \mathcal{Z}(\mathcal{F}(E)) = \mathcal{F}(\mathcal{Z}(E)) = V(J)$.

Proof. (i): Suppose $E = V(I)$ for some ideal I of R . We claim that $\mathcal{F}(E) = \text{Im } \pi^*$ where $\pi : R \rightarrow S^{-1}R$ is the canonical map with $S = 1 + I$. The inclusion $\mathcal{F}(E) \subseteq \text{Im } \pi^*$ is obvious. To prove the reverse inclusion, let \mathfrak{q} be a prime ideal of R such that $\mathfrak{q} \cap S = \emptyset$. There exists a prime ideal \mathfrak{p} of R such that $\mathfrak{q} \subseteq \mathfrak{p}$ and the extended ideal $S^{-1}\mathfrak{p}$ is a maximal ideal of $S^{-1}R$. We have $I \subseteq \mathfrak{p}$. If not, then choose some element $x \in I \setminus \mathfrak{p}$. Clearly $(\mathfrak{p} + Rx) \cap S \neq \emptyset$. Thus there are elements $r \in R$ and $y \in I$ such that $1 + rx + y \in \mathfrak{p}$. But this is a contradiction since $\mathfrak{p} \cap S = \emptyset$.

(ii): Suppose $E = \text{Im } \varphi^*$ for some ring homomorphism $\varphi : R \rightarrow A$. Then $\mathcal{Z}(E) = V(I)$ where $I = \text{Ker } \varphi$. Because the inclusion $\mathcal{Z}(E) \subseteq V(I)$ is obvious. To prove the reverse inclusion, pick $\mathfrak{p} \in V(I)$. Let \mathfrak{q} be a minimal prime of I such that $\mathfrak{q} \subseteq \mathfrak{p}$. By [10, Lemma 3.9], $\mathfrak{q} \in E$.

(iii): By Lemma 4.1, it suffices to show that $\text{Ann}_R(a) + J = R$ for all $a \in J$. Suppose there is some $b \in J$ such that $\text{Ann}_R(b) + J \subseteq \mathfrak{p}$ where \mathfrak{p} is a prime ideal of R . There is some $x \in I$ such that $(1 + x)b = 0$ and

so $1 + x \in \mathfrak{p}$. Furthermore $\mathcal{Z}(\mathcal{F}(E)) = V(J)$. Therefore there exists a prime ideal $\mathfrak{q} \in \mathcal{F}(E)$ such that $\mathfrak{q} \subseteq \mathfrak{p}$. It follows that $I \subseteq \mathfrak{p}$ since E is stable under the generalization. But this is a contradiction and we win. \square

The following result provides various characterizations for S-rings:

Theorem 4.3. *The following conditions are equivalent.*

- (i) *The ring R is an S-ring.*
- (ii) *Every cyclic flat R -module is R -projective.*
- (iii) *Every flat R -module of the form R/I is R -projective where I is an ideal of R .*
- (iv) *Every Zariski closed subset of $\text{Spec}(R)$ which is stable under the generalization is Zariski open.*
- (v) *Every patch closed subset of $\text{Spec}(R)$ which is stable under the generalization and specialization is patch open.*
- (vi) *Every flat closed subset of $\text{Spec}(R)$ which is stable under the specialization is flat open.*
- (vii) *Each double-closed subset of $\text{Spec}(R)$ is of the form $V(e)$ where $e \in R$ is an idempotent element.*
- (viii) *For every sequence $(x_n)_{n \geq 1}$ of elements of R if $x_n = x_n x_{n+1}$ for all n then there exists some k such that x_k is an idempotent and $x_n = x_k$ for all $n \geq k$.*
- (ix) *For every sequence $(y_n)_{n \geq 1}$ of elements of R if $y_{n+1} = y_n y_{n+1}$ for all n then there exists some k such that y_k is an idempotent and $y_n = y_k$ for all $n \geq k$.*

Proof. The implications (i) \Rightarrow (ii) and (ii) \Rightarrow (iii) are obvious.
 (iii) \Rightarrow (iv) : Suppose $E \subseteq \text{Spec}(R)$ is stable under the generalization and $E = V(I)$ for some ideal I of R . By Theorem 4.2, there is an ideal J such that R/J is R -flat and $E = V(J)$. Thus, by the hypothesis, R/J is R -projective and so by Lemma 3.1, E is Zariski open.
 (iv) \Leftrightarrow (v) \Leftrightarrow (vi) : See [10, Theorem 3.11].
 (vi) \Rightarrow (vii) : Apply [10, Theorem 3.11] and [3, Tag 00EE].
 (vii) \Rightarrow (iv) : There is nothing to prove.
 (iv) \Rightarrow (i) : Let M be a f.g. flat R -module. To prove the assertion, by Theorem 3.8, it suffices to show that for each natural number n , $\psi^{-1}(\{n\})$ is Zariski open where ψ is the rank map of M , see [12, Remark 2.3]. By [12, Lemma 4.1], $\psi^{-1}(\{n\}) = \text{Supp } N \cap (\text{Spec}(R) \setminus \text{Supp } N')$ where $N = \Lambda^n(M)$ and $N' = \Lambda^{n+1}(M)$. But $\text{Supp } N$ and $\text{Supp } N'$ are Zariski closed since by [12, Lemma 6.3], N and N' are f.g. R -modules.

By [12, Lemma 6.6], N is a flat R -module. Thus, by [12, Corollary 3.2] and by the hypothesis, $\text{Supp } N$ is Zariski open. Therefore $\psi^{-1}(\{n\})$ is Zariski open.

(iii) \Rightarrow (viii) : Let $I = \langle x_n : n \geq 1 \rangle$. By Lemma 4.1, R/I is R -flat and so, by the hypothesis, it is R -projective. Thus, by Lemma 3.1, there is an element $c \in I$ such that $I = Rc$. It follows that there is some $d \geq 1$ such that $Rc = Rx_d$ since $I = \bigcup_{n \geq 1} Rx_n$. Let $k = d + 1$.

There exists some $r \in R$ such that $x_k = rx_d = rx_dx_k = x_k^2$. We also have $x_{k+1} = r'x_k$ for some $r' \in R$. It follows that $x_{k+1} = x_{k+1}x_k = x_k$ and by the induction we obtain that $x_n = x_k$ for all $n \geq k$.

(viii) \Rightarrow (iii) : Let I be an ideal of R such that R/I is R -flat. We shall prove that I is generated by an idempotent element. To do this we act as follows. Let \mathcal{I} be the set of ideals of the form Re where $e \in I$ is an idempotent element. Let $\{Re_n : n \geq 1\}$ be an ascending chain of elements of \mathcal{I} . For each n there is some $r_n \in R$ such that $e_n = r_n e_{n+1}$. It follows that $e_n = e_n e_{n+1}$. Thus, by the hypothesis, the chain $Re_1 \subseteq Re_2 \subseteq \dots$ is stationary. Therefore, by the axiom of choice, \mathcal{I} has at least a maximal element Re . We also claim that if $J = \langle z_n : n \geq 1 \rangle$ is a countably generated ideal of R with $J \subseteq I$ then there is an idempotent $e' \in I$ such that $J \subseteq Re'$. By Lemma 4.1, there is an $x_1 \in I$ such that $z_1 = z_1 x_1$. Then for the pair (x_1, z_2) , again by Lemma 4.1, we may find an $x_2 \in I$ such that $x_1 = x_1 x_2$ and $z_2 = z_2 x_2$. Therefore, in this way, we obtain a sequence (x_n) of elements of I such that $J \subseteq L = \langle x_n : n \geq 1 \rangle$ and $x_n = x_n x_{n+1}$ for all $n \geq 1$. But, by the hypothesis, there exists some $k \geq 1$ such that x_k is an idempotent and $x_n = x_k$ for all $n \geq k$. It follows that $L = Rx_k$. This establishes the claim. Now pick $x \in I$. Then, by what we have just proved, there is an idempotent $e' \in I$ such that $Re \subseteq \langle e, x \rangle \subseteq Re'$. By the maximality of Re , we obtain that $e = e'$. Thus $I = Re$ and so R/I as R -module is isomorphic to $R(1 - e)$. Therefore R/I is R -projective.

(viii) \Leftrightarrow (ix) : Let (x_n) be a sequence of elements of R . Put $y_n = 1 - x_n$ for all n . Then $x_n = x_n x_{n+1}$ if and only if $y_{n+1} = y_n y_{n+1}$. \square

5. EXPLORING THE PROJECTIVITY OF F.G. FLAT MODULES

As a consequence of Theorem 4.3, we obtain the following result which in turn vastly generalizes some previous results in the literature specially including [6, Theorem 4.38], [4, Corollary 1.5], [8, Fact 7.5] and [9, Corollary 3.57] in the commutative case.

Theorem 5.1. *Let R be a ring which has either a finitely many minimal primes or a finitely many maximal ideals. Then R is an S -ring.*

Proof. Let F be a patch closed subset of $\text{Spec}(R)$ which is stable under the generalization and specialization. By Theorem 4.3, it suffices to show that it is a patch open. First assume that $\text{Min}(R) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. There exists some s with $1 \leq s \leq n$ such that $\mathfrak{p}_s, \mathfrak{p}_{s+1}, \dots, \mathfrak{p}_n \notin F$ but $\mathfrak{p}_i \in F$ for all $i < s$. It follows that $\text{Spec}(R) \setminus F = \bigcup_{i=s}^n V(\mathfrak{p}_i)$. Therefore F is Zarsiki open in this case and so it is patch open. Now assume that $\text{Max}(R) = \{\mathfrak{m}_1, \dots, \mathfrak{m}_d\}$. Similarly, there exists some k with $1 \leq k \leq d$ such that $\mathfrak{m}_k, \mathfrak{m}_{k+1}, \dots, \mathfrak{m}_d \notin F$ but $\mathfrak{m}_i \in F$ for all $i < k$. We have $\text{Spec}(R) \setminus F = \bigcup_{i=k}^d \Lambda(\mathfrak{m}_i)$. Therefore, by [10, Corollary 3.6], F is a flat open in this case and so it is patch open. \square

Remark 5.2. Every projective module over a local ring is free, see [3, Tag 0593]. But in general this is not necessarily true even for a semi-local ring (a ring with a finitely many maximal ideals) which is not local. As a specific example, let $n > 1$ be a natural number with at least two distinct prime factors and let $n = p_1^{s_1} \dots p_k^{s_k}$ be its prime factorization where the p_i are distinct prime numbers and $s_i \geq 1$ for all i . Each A_i can be considered as R -module through the canonical ring map $R \rightarrow A_i$ where $R = \mathbb{Z}/n\mathbb{Z}$ and $A_i = \mathbb{Z}/p_i^{s_i}\mathbb{Z}$. By the Chinese remainder theorem, see [3, Tag 00DT], R as module over itself is isomorphic to the direct sum $\bigoplus_{i=1}^k A_i$. Thus each A_i is R -projective. But none of them is R -free since every non-zero free R -module has at least n elements while $p_i^{s_i} < n$ for all i . Note that R is a semi-local ring with the maximal ideals $p_i\mathbb{Z}/n\mathbb{Z}$ where $1 \leq i \leq k$.

Under the light of Theorems 3.9 and 3.11, a stronger result is obtained:

Theorem 5.3. *Let $\varphi : A \rightarrow B$ be a ring map whose kernel is contained in the radical Jacobson of A . Then the following hold.*

- (i) *If M is a f.g. flat A -module such that $M \otimes_A B$ is B -projective then M is A -projective.*
- (ii) *If B is an S -ring then A itself is an S -ring.*

Proof. (i): Clearly M/JM is a f.g. flat A/J -module and $M/JM \otimes_{A/J} B \simeq M \otimes_A B$ is B -projective where $J = \text{Ker } \varphi$. Moreover A/J can be viewed as a subring of B via φ . Therefore, by Theorem 3.9, M/JM is A/J -projective. Then apply Theorem 3.11.

(ii): It implies from (i). Because if M is a f.g. flat A -module then $M \otimes_A B$ is a f.g. flat B -module and so, by the hypothesis, it is B -projective. Thus by (i), M is A -projective. \square

Theorem 5.3(i) generalizes Theorem 3.9, Theorem 3.11 and [1, Proposition 2.3] while its second part generalizes [2, Theorem 2], [4, Theorem 2.2] and [1, Theorem 2.4(i)] moreover it generalizes [8, Proposition 5.5 and Corollary 5.6] in the commutative case.

Remark 5.4. Let S be a subset of a ring R . The polynomial ring $R[x_s : s \in S]$ modulo I is denoted by $S^{(-1)}R$ where the ideal I is generated by elements of the form $sx_s^2 - x_s$ and $s^2x_s - s$ with $s \in S$. We call $S^{(-1)}R$ the pointwise localization of R with respect to S . Amongst them, the pointwise localization of R with respect to itself, namely $R^{(-1)}R$, has more interesting properties; for further information please consult with [7] also see [15]. Note that Wiegand, in [14] and [15], utilizes the notation \widehat{R} instead of $R^{(-1)}R$. It is easy to see that the pair $(S^{(-1)}R, \eta)$ where $\eta : R \rightarrow S^{(-1)}R$ is the canonical map satisfies in the following universal property: “if $\varphi : R \rightarrow A$ is a ring map with the property that for each $s \in S$ there is an element $c \in A$ for which $\varphi(s) = \varphi(s)^2c$ and $c = \varphi(s)c^2$ then there exists a unique ring map $\psi : S^{(-1)}R \rightarrow A$ such that $\varphi = \psi \circ \eta$.” Now let \mathfrak{p} be a prime ideal of R and consider the canonical map $\pi : R \rightarrow \kappa(\mathfrak{p})$ where $\kappa(\mathfrak{p})$ is the residue field of R at \mathfrak{p} . By the above universal property, there is a ring map $\psi : S^{(-1)}R \rightarrow \kappa(\mathfrak{p})$ such that $\pi = \psi \circ \eta$. Thus η induces a surjection between the corresponding spectra. This, in particular, implies that the kernel of η is contained in the nil-radical of R . Using this, then the following result vastly generalizes [14, Theorem 2].

Corollary 5.5. *Let M be a f.g. flat R -module. If there exists a subset S of R such that $M \otimes_R S^{(-1)}R$ is $S^{(-1)}R$ -projective then M is R -projective.*

Proof. It is an immediate consequence of Theorem 5.3(i). \square

Example 5.6. Here we give an example of a f.g. flat module which is not projective. Let $R = \prod_{i \geq 1} A$ be an infinite direct product of copies of a non-zero ring A and let $I = \bigoplus_{i \geq 1} A$ which is an ideal of R . Let $x = (x_i)$ be an element of I . There exists a finite subset D of $\{1, 2, 3, \dots\}$ such that $x_i = 0$ for all $i \in \{1, 2, 3, \dots\} \setminus D$. Now consider the sequences $y = (y_i)$ and $z = (z_i)$ of elements of R with $y_i = 0$ and $z_i = 1$ for all $i \in D$ moreover $y_i = 1$ and $z_i = 0$ for all $i \in \{1, 2, 3, \dots\} \setminus D$. Clearly $y \in \text{Ann}_R(x)$, $z \in I$ and $1_R = y + z$. Thus, by Lemma 4.1, R/I is R -flat. Suppose R/I is R -projective. Then, by Lemma 3.1, there is a sequence $e = (e_i) \in R$ such that $I = Re$. Thus there exists a finite subset E of $\{1, 2, 3, \dots\}$ such that $e_i = 0$ for all $i \in \{1, 2, 3, \dots\} \setminus E$. Clearly $\{1, 2, 3, \dots\} \setminus E \neq \emptyset$. Pick some $k \in \{1, 2, 3, \dots\} \setminus E$. There is some $r = (r_i) \in R$ such that $(\delta_{i,k})_{i \geq 1} = re$. In particular, $1_A = r_k e_k = r_k 0_A = 0_A$. This is a contradiction. Therefore R/I is not R -projective. This observation leads us to the following result.

Proposition 5.7. *A direct product of rings is an S-ring if and only if the index set of the product is a finite set and all of the components are S-rings.*

Proof. Let $R = \prod_{i \in I} R_i$ be an S-ring. We may assume that all of the rings R_i are non-zero. Suppose I is an infinite set. Consider a well-ordered relation $<$ on I . Let i_1 be the least element of I and for each natural number $n \geq 1$, by induction, let i_{n+1} be the least element of $I \setminus \{i_1, \dots, i_n\}$. Now we define $x_n = (r_{n,i})_{i \in I}$ as an element of R by $r_{n,i} = 1$ for all $i \in \{i_1, \dots, i_n\}$ and $r_{n,i} = 0$ for all $i \in I \setminus \{i_1, \dots, i_n\}$. Clearly the sequence (x_n) satisfies the condition $x_n = x_n x_{n+1}$. Thus, by Theorem 4.3, there is some k such that $x_n = x_k$ for all $n \geq k$. But this is a contradiction. Thus I should be a finite set. The remaining statements, by applying Theorem 4.3(viii), are straightforward. \square

Sometimes a ring would be a S-ring even if it has infinitely many minimal primes and infinitely many maximal ideals. More precisely:

Theorem 5.8. *Let X be a subset of $\text{Spec}(R)$ with the property that for each maximal ideal \mathfrak{m} of R there exists some $\mathfrak{p} \in X$ such that $\mathfrak{p} \subseteq \mathfrak{m}$. If the collection of subsets $X \cap V(f)$ where f runs through the underlying set of R satisfies either the ascending chain condition or the descending*

chain condition then R is an S-ring.

Proof. By Theorem 5.3(ii), it suffices to show that R/J is an S-ring where $J = \bigcap_{\mathfrak{p} \in X} \mathfrak{p}$. Let (x_n) be a sequence of elements of R/J such that $x_n = x_n x_{n+1}$ for all n . Suppose $x_n = a_n + J$ for all n . Let $E_n = X \cap V(a_n)$ and let $F_n = X \cap V(1 - a_n)$. Clearly $E_n \supseteq E_{n+1}$, $F_n \subseteq F_{n+1}$ and $X = E_n \cup F_{n+1}$. First assume the descending chain condition. Then there exists some $d \geq 1$ such that $E_n = E_d$ for all $n \geq d$. Therefore $X = E_n \cup F_n$ for all $n > d$. Thus $a_n(1 - a_n) \in \mathfrak{p}$ for all $\mathfrak{p} \in X$ and all $n > d$. It follows that $x_n = x_n^2$ for all $n > d$. The chain $\langle x_{d+1} \rangle \subseteq \langle x_{d+2} \rangle \subseteq \dots$ eventually stabilizes. If not, then the ascending chain $V(1 - x_{d+1}) \subseteq V(1 - x_{d+2}) \subseteq \dots$ does not stabilize. Therefore we may find some $k > d$ such that $V(1 - x_k)$ is a proper subset of $V(1 - x_{k+1})$. Thus there exists a prime ideal \mathfrak{q} of R such that $J \subseteq \mathfrak{q}$ and $1 - a_{k+1}, a_k \in \mathfrak{q}$. There is a maximal ideal \mathfrak{m} of R such that $\mathfrak{q} \subseteq \mathfrak{m}$. By the hypotheses, there is a $\mathfrak{p} \in X$ such that $\mathfrak{p} \subseteq \mathfrak{m}$. Clearly $1 - a_{k+1}, a_k \in \mathfrak{p}$. This means that E_{k+1} is a proper subset of E_k . But this is a contradiction. Thus there is some $s > d$ such that $\langle x_n \rangle = \langle x_s \rangle$ for all $n \geq s$. Therefore $x_n = x_s$ for all $n \geq s$ and so by Theorem 4.3, R/J is an S-ring in the case of the descending chain condition. Apply a similar argument as above for the chain $F_1 \subseteq F_2 \subseteq \dots$ in the case of the ascending chain condition. \square

Corollary 5.9. *If the collection of subsets $\text{Min}(R) \cap V(f)$ where $f \in R$ satisfies either the ascending chain condition or the descending chain condition then R is an S-ring.*

Proof. It implies from Theorem 5.8 by taking $X = \text{Min}(R)$. \square

Theorem 5.8 vastly generalizes [8, Proposition 7.6]:

Corollary 5.10. *If the collection of subsets $\text{Max}(R) \cap V(f)$ where $f \in R$ satisfies either the ascending chain condition or the descending chain condition then R is an S-ring.*

Proof. In Theorem 5.8, put $X = \text{Max}(R)$. \square

The author believes that still can be found some other results in the literature on the projectivity of f.g. flat modules which can be easily

deduced them by using the results of this article.

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